

BROWNIAN MOTION IN A BALL IN THE PRESENCE OF SPHERICAL OBSTACLES

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ABSTRACT. We study the problem of when a Brownian motion in the unit ball has a positive probability of avoiding a countable collection of spherical obstacles. We give a necessary and sufficient integral condition for such a collection to be avoidable.

1. INTRODUCTION

The setting in this paper is the unit ball, $\mathbb{B} = \{x \in \mathbb{R}^d : |x| < 1\}$, in Euclidean space \mathbb{R}^d where $d \geq 3$. We study the problem of when Brownian motion in the ball has a positive probability of avoiding a countable collection of spherical obstacles and thereby reaching the outer boundary of \mathbb{B} .

We denote by Λ a sequence of points in \mathbb{B} . To each point λ in this sequence we associate a spherical obstacle, $B(\lambda, r_\lambda)$, where

$$B(\lambda, r_\lambda) = \{x : |\lambda - x| \leq r_\lambda\},$$

and denote by $\partial B(\lambda, r_\lambda)$ the boundary of this obstacle. We let \mathcal{B} denote the countable collection of closed spherical obstacles,

$$\mathcal{B} = \bigcup_{\lambda \in \Lambda} B(\lambda, r_\lambda).$$

We assume that the spherical obstacles are pairwise disjoint, lie inside the ball \mathbb{B} and that the origin lies outside \mathcal{B} . We call a collection of spherical obstacles *avoidable* if there is positive probability that Brownian motion, starting from the origin, hits the boundary of \mathbb{B} before hitting any of the spherical obstacles in \mathcal{B} . This is equivalent to positive harmonic measure at 0 of the boundary of the unit ball with respect to the domain $\Omega = \mathbb{B} \setminus \mathcal{B}$, consisting of the unit ball less the obstacles, that is $\omega(0, \partial\mathbb{B}; \Omega) > 0$.

In the setting of the unit disk, Ortega-Cerdà and Seip [7] gave an integral condition for a collection of disks to be avoidable. In [5], Carroll and Ortega-Cerdà gave an integral criterion for a configuration of balls in \mathbb{R}^d , $d \geq 3$, to be avoidable. Thus, it seems natural to ask if Ortega-Cerdà and Seip's result for the disk in the plane can be extended to the ball in space. A solution to this problem is the main result of this paper.

Next, we put some restrictions on the spacing of the spherical obstacles. A collection of obstacles, \mathcal{B} , is *regularly spaced* if it is separated, in that there exists $\epsilon > 0$ such that given any $\lambda, \lambda' \in \Lambda$ with $|\lambda| \geq |\lambda'|$, then $|\lambda - \lambda'| > \epsilon(1 - |\lambda|)$; uniformly dense, in that there exists R with $0 < R < 1$ such that for $x \in \mathbb{B}$, the

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ball $B(x, R(1 - |x|))$ contains at least one $\lambda \in \Lambda$; and finally the radius $r_\lambda = \phi(|\lambda|)$ where $\phi : [0, 1] \rightarrow [0, 1]$ is a decreasing function.

Answering a question of Akeroyd in [2], Ortega-Cerdà and Seip [7] proved the following theorem.

Theorem A. A collection of regularly spaced disks in the unit disk is avoidable if and only if

$$\int_0^1 \frac{dt}{(1-t) \log((1-t)/\phi(t))} < \infty.$$

This theorem in [7] is expressed in terms of pseudo-hyperbolic disks. We extend Theorem A to the setting of the unit ball in \mathbb{R}^d , $d \geq 3$.

Theorem 1.1. *The collection of regularly spaced closed spherical obstacles \mathcal{B} in \mathbb{B} is avoidable if and only if*

$$(1.1) \quad \int_0^1 \frac{\phi(t)^{d-2}}{(1-t)^{d-1}} dt < \infty.$$

We present two proofs of this result. The first proof exploits a connection between avoidability and minimal thinness, a potential theoretic measure of the size of a set near a boundary point of a region. We learnt of this from both the paper of Lundh [6] and from Professor S.J. Gardiner. We also make use of a Wiener-type criterion for minimal thinness due to Aikawa [1].

The second proof is more direct and transparent. It is an adaptation of Ortega-Cerdà and Seip's proof of Theorem A in [7], the key difference being that in higher dimensions we do not have the luxury of conformal mapping.

2. AVOIDABLE OBSTACLES AND MINIMAL THINNESS

Following the notation of Lundh [6], we let $SH(\mathbb{B})$ denote the class of non-negative superharmonic functions on the unit ball and let P_τ denote the Poisson kernel at $\tau \in \partial\mathbb{B}$. For a positive superharmonic function h on \mathbb{B} the reduced function of h with respect to a subset E of \mathbb{B} is

$$R_h^E(w) = \inf\{u(w) : u \in SH(\mathbb{B}), u(x) \geq h(x), x \in E\}$$

and the regularized reduced function $\widehat{R}_h^E(w) = \liminf_{x \rightarrow w} R_h^E(x)$.

Definition 2.1. A set E is *minimally thin* at $\tau \in \partial\mathbb{B}$ if there is an x_0 in the unit ball such that $\widehat{R}_{P_\tau}^E(x_0) < P_\tau(x_0)$.

A nice account of reduced functions and minimal thinness may be found in [3, Page 38 ff] or [4, Chapter 9].

2.1. Avoidability and minimal thinness. Lundh proves the following result in [6]. We include a brief proof for the convenience of the reader.

Proposition 2.2. Let A be a closed subset of \mathbb{B} such that $\mathbb{B} \setminus A$ contains the origin and is connected. Let $\mathcal{M} = \{\tau \in \partial\mathbb{B} : A \text{ is minimally thin at } \tau\}$. Then the following are equivalent:

- A is avoidable,
- $|\mathcal{M}| > 0$,

where $| \cdot |$ denotes surface area on the unit ball.

Proof. Noting that

$$1 = \int_{\partial\mathbb{B}} P_\tau(x) \frac{d\tau}{|\partial\mathbb{B}|},$$

and taking $h \equiv 1$ in [4, Corollary 9.1.4], we see that

$$\widehat{R}_1^A(x) = \int_{\partial\mathbb{B}} \widehat{R}_{P_\tau}^A(x) \frac{d\tau}{|\partial\mathbb{B}|}.$$

Also, it follows from [3, Page 653, 14.3sm] that the regularized reduced function of 1 with respect to A evaluated at x is the harmonic measure at x of ∂A in the domain $\mathbb{B} \setminus A$. Thus,

$$\omega(0, \partial A, \mathbb{B} \setminus A) = \widehat{R}_1^A(0) = \frac{1}{|\partial\mathbb{B}|} \int_{\partial\mathbb{B}} \widehat{R}_{P_\tau}^A(0) d\tau.$$

Since $\widehat{R}_{P_\tau}^A(0) \leq P_\tau(0) = 1$, it follows that $\omega(0, \partial A, \mathbb{B} \setminus A) < 1$ if and only if the set $\mathcal{M}_0 = \{\tau \in \partial\mathbb{B}, \widehat{R}_{P_\tau}^A(0) < 1\}$ has positive measure. In the connected domain $\mathbb{B} \setminus A$, the set \mathcal{M}_0 is the same as the set \mathcal{M} . Thus, A being avoidable, that is $\omega(0, \partial\mathbb{B}; \mathbb{B} \setminus A) > 0$, is equivalent to \mathcal{M} having positive measure. \square

2.2. Minimal thinness and a Wiener-type criterion. It is a standard result, see for example Aikawa [1] or Lundh [6], that a set is minimally thin at a point if and only if it satisfies a Wiener-type criterion. Let $\{Q_k\}$ be a Whitney decomposition of the unit ball \mathbb{B} in \mathbb{R}^d ($d \geq 3$) and let q_k be the Euclidean distance from the centre, c_k , of the Whitney cube Q_k to the boundary of \mathbb{B} . Let A be a subset of \mathbb{B} . Let τ be a boundary point of \mathbb{B} and $\rho_k(\tau)$ be the distance from c_k to the boundary point τ . Let cap denote Newtonian capacity. Then A is minimally thin at the point τ if and only if

$$(2.1) \quad \sum_k \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(A \cap Q_k) < \infty.$$

In the next section, we consider this Wiener-type criterion in the particular setting of the unit ball less a collection of regularly spaced spherical obstacles.

2.3. Wiener-type criterion and integral condition. For a constant $K > 1$, we let $S_j = \{x : |x| = 1 - K^{-j}\}$ be the sphere of radius $\rho_j = 1 - K^{-j}$ and B_j be the interior of this sphere. We denote by A_j the annulus bounded by S_j and S_{j-1} , and write ϕ_j for $\phi(\rho_j)$.

Proposition 2.3. Let \mathcal{B} be a regularly spaced collection of spherical obstacles in \mathbb{B} .

- (i) If the set \mathcal{B} satisfies the Wiener-type criterion (2.1) at some point in $\partial\mathbb{B}$ then the integral condition (1.1) holds,
- (ii) The integral condition (1.1) implies that \mathcal{B} satisfies the Wiener-type criterion (2.1) at all points $\tau \in \partial\mathbb{B}$.

Proof. We first assume that the integral condition holds and we'll show that (2.1) follows. We note that the integral condition (1.1) is equivalent to

$$(2.2) \quad \sum_{j=1}^{\infty} (\phi_j K^j)^{d-2} < \infty,$$

where $K > 1$. By the separation condition on the sequence Λ , there is an N such that any cube Q_k can contain no more than N points in Λ . Splitting the sum in (2.1) into a sum over annuli we obtain

$$(2.3) \quad \sum_k \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(\mathcal{B} \cap Q_k) = \sum_{j=1}^{\infty} \sum_{k:c_k \in A_j} \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(\mathcal{B} \cap Q_k)$$

$$(2.4) \quad \leq \sum_{j=1}^{\infty} N(K^{-j})^2 \phi_j^{d-2} \sum_{k:c_k \in A_j} \frac{1}{\rho_k(\tau)^d},$$

since the capacity of a ball with radius ϕ_j is equal to ϕ_j^{d-2} . We now concentrate on the latter sum in (2.4). We split up the j^{th} annulus A_j into rings centered at the projection of τ onto the sphere S_j , and with radius equal to nK^{-j} where we recall that K^{-j} is the distance from τ to S_j . There are at most

$$\frac{c_d(nK^{-j})^{d-2}}{(K^{-j})^{d-2}} = c_d n^{d-2}$$

Whitney cubes in each ring where c_d is a constant depending on the dimension, d . For the n^{th} ring,

$$\rho_k(\tau) \geq nK^{-j}$$

and N_j rings intersect the annulus A_j . Thus,

$$\begin{aligned} \sum_{k:c_k \in A_j} \frac{1}{\rho_k(\tau)^d} &\leq \sum_{n=1}^{N_j} \frac{c_d n^{d-2}}{(nK^{-j})^d} \\ &\leq (K^j)^d c_d \sum_{n=1}^{N_j} \frac{1}{n^2}. \end{aligned}$$

Thus, we see that the Wiener-type series (2.3) is convergent.

We now assume that the set \mathcal{B} satisfies (2.1) at some arbitrary point $\tau \in \partial\mathbb{B}$ and show that this implies the integral condition (1.1). We choose K sufficiently large so that for all j bigger than a fixed constant there is at least one centre of a ball in each Whitney cube, Q_k , in the resulting Whitney decomposition of \mathbb{B} . Starting with the Wiener-type series we split it into a sum over the annuli A_j and then proceed to ignore all Whitney cubes in A_j except one near to the point τ , for which $\rho_k(\tau) \leq K^{-j}$, as follows.

$$\begin{aligned} \sum_k \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(\mathcal{B} \cap Q_k) &= \sum_{j=1}^{\infty} \sum_{k:c_k \in A_j} \frac{q_k^2}{\rho_k(\tau)^d} \text{cap}(\mathcal{B} \cap Q_k) \\ &\geq \sum_{j=0}^{\infty} K^{-2j} \phi_j^{d-2} \frac{1}{\rho_k(\tau)^d} \\ &\geq \sum_{j=0}^{\infty} (\phi_j K^j)^{d-2} \end{aligned}$$

Thus, since the Wiener-type series is convergent, (2.2) follows and so the integral condition (1.1) holds. \square

Combining Proposition 2.2, the Wiener-type criterion (2.1) and Proposition 2.3 we have a proof of Theorem 1.1. We note that the method used in this section could also be used to give an alternative proof of Ortega-Cerdà and Seip's Theorem A.

3. DIRECT PROOF OF THEOREM 1.1

We now give an alternative proof of Theorem 1.1 by adapting the method of Ortega-Cerdà and Seip in [7]. In dimensions higher than 2 we do not have conformal mapping, but we do have the Kelvin transform. We let

$$x^* = \frac{\rho_{j+1}^2}{|x|^2} x$$

be the inversion of the point x in the sphere of radius ρ_{j+1} . We note that $|x||x^*|$ equals ρ_{j+1}^2 , and let $\phi(|\lambda|) = \phi_\lambda$. We begin with a lemma, prove the sufficiency of the integral condition in the next subsection and the necessity in the following one.

Lemma 3.1. *Let $K > \max\{4, \frac{1+R}{1-R}\}$ and x be an arbitrary point belonging to S_{j-1} . There is a centre of an obstacle, $\lambda_x \in \Lambda$, such that λ_x lies in the annulus A_j bounded by S_{j-1} and S_j , and*

$$|x - \lambda_x| \leq \frac{K-1}{K} |x^* - \lambda_x|.$$

Proof. For $x \in S_{j-1}$, let x' be the point on the extension of the radius of S_j containing x , and located halfway between S_{j-1} and S_j . Then x' is a distance $K^{-(j-1)} - \frac{K-1}{2K^j}$ from the boundary of the ball \mathbb{B} . Since Λ is uniformly dense, the ball $B(x', R(1 - |x'|))$ contains some $\lambda_x \in \Lambda$. Also, due to the choice of K , the ball $B(x', R(1 - |x'|))$ is contained in the annulus A_j . Let x'' be on the same ray as x and x^* and also on S_j . We first note that $|x - \lambda_x| \leq |x - x''|$ and $|x^* - \lambda_x| > |x^* - x''|$. Also, we note that $|x| = \rho_{j-1}$, $|x''| = \rho_j$ and $|x^*| = \rho_{j+1}^2 / \rho_{j-1}$. Thus,

$$(3.1) \quad |x - \lambda_x| \leq |x - x''| = (K-1)K^{-j}.$$

Also,

$$|x^* - \lambda_x| \geq |x^* - x''| = \frac{(1 - K^{-(j+1)})^2}{1 - K^{-(j-1)}} - (1 - K^{-j}) \geq K^{-j+1},$$

for $j \geq 2$. Thus,

$$|x - \lambda_x| \leq \frac{K-1}{K} |x^* - \lambda_x|,$$

as required. \square

3.1. Integral Condition (1.1) implies avoidability. We first assume (1.1) and show that the spherical obstacles are avoidable that is, we show that $\omega(0, \partial\mathbb{B}; \Omega) > 0$. We split the collection of spherical obstacles into those with centres inside and those with centres outside a ball of radius $r < 1$. We let $\Lambda_r = \{\lambda \in \Lambda : |\lambda| > r\}$ and let

$$\mathcal{B}_r = \bigcup_{\lambda \in \Lambda_r} B(\lambda, r_\lambda) = \bigcup_{\lambda \in \Lambda_r} B_\lambda$$

denote the infinitely many spherical obstacles with centres outside $B(0, r)$. Also, we let $\Omega_r = \mathbb{B} \setminus \mathcal{B}_r$ be the champagne subregion where all obstacles have centres outside a ball of radius r . We may safely ignore the finitely many spherical obstacles with centres inside the ball of radius r . Thus, it is sufficient to show that $\omega(0, \partial\mathbb{B}; \Omega_r) > 0$

for some r with $0 < r < 1$, which is equivalent to showing that $\omega(0, \partial\mathcal{B}_r; \Omega_r) < 1$. We choose r such that

$$\int_r^1 \frac{\phi(t)^{d-2}}{(1-t)^{d-1}} dt < \frac{\epsilon^d (K-1)^{d-2}}{2^{d+1} d(d-2) K^{2d-1}}$$

and let n_r be the biggest integer smaller than $1 + \log(\frac{1}{1-r})/\log K$. This ensures that $r > \lfloor 1 - K^{-(n_r-1)} \rfloor$. We proceed as follows,

$$\begin{aligned} \omega(0, \partial\mathcal{B}_r; \Omega_r) &= \sum_{\lambda \in \Lambda_r} \omega(0, \partial B_\lambda; \Omega_r) \leq \sum_{\lambda \in \Lambda_r} \omega(0, \partial B_\lambda; \mathbb{B} \setminus B_\lambda) \\ &\leq \sum_{j=n_r}^{\infty} \left(\sum_{\lambda \in A_j} \omega(0, \partial B_\lambda; \mathbb{B} \setminus B_\lambda) \right). \end{aligned}$$

We now obtain an upper bound for the number of centres in A_j and an upper bound for the contribution of an obstacle with centre in A_j to the above sum. Due to the separation condition, centres of balls in A_j are at least ϵK^{-j} apart. Thus, the number of centres in A_j , which is less than the volume of A_j divided by the volume of a ball with radius $\epsilon K^{-j}/2$, is less than

$$\frac{2^d d K^2}{\epsilon^d} K^{(d-1)j}.$$

Next, we want an upper bound for $\omega(0, \partial B_\lambda; \mathbb{B} \setminus B_\lambda)$. We construct a suitable function h that is harmonic on $\mathbb{B} \setminus B_\lambda$, continuous on its closure and also satisfies $h(x) \geq 1$, $x \in \partial B_\lambda$ and $h(x) \geq 0$, $x \in \partial \mathbb{B}$. Then, using the Maximum Principle, we obtain the required upper bound. Consider the function

$$h(x) = 2 [u_\lambda(x) - u_\lambda^*(x)],$$

where

$$u_\lambda(x) = \left[\frac{\phi_\lambda}{|x - \lambda|} \right]^{d-2}, \quad u_\lambda^*(x) = \left[\frac{\phi_\lambda}{|x||x^* - \lambda|} \right]^{d-2} \quad \text{and} \quad x^* = \frac{1}{|x|^2} x.$$

We note that u_λ and u_λ^* are harmonic. Also, $1/2$ is a lower bound for $u_\lambda(x) - u_\lambda^*(x)$ for $x \in \partial B_\lambda$ which we show as follows. For $x \in \partial B_\lambda$, we have that $|x| \geq 1 - K^{-1}$ and $|x^* - \lambda| \geq K^{-j}$, hence

$$u_\lambda(x) - u_\lambda^*(x) = 1 - \left[\frac{\phi_\lambda}{|x||x^* - \lambda|} \right]^{d-2} \geq 1 - \left[\frac{K\phi_{j-1}}{(K-1)K^{-j}} \right]^{d-2}.$$

It follows from (2.2) that

$$\lim_{j \rightarrow \infty} \frac{\phi_{j-1}}{K^{-j}} = 0.$$

Thus, there exists N such that for $j > N$

$$u_\lambda(x) - u_\lambda^*(x) > \frac{1}{2}.$$

Thus, $h(x)$ satisfies the required criteria and is an upper bound for the harmonic measure $\omega(x, \partial B_\lambda; \mathbb{B} \setminus B_\lambda)$.

Next, we want an upper estimate for $h(0)$. We first note that as $x \rightarrow 0$, $x^* \rightarrow \infty$ and also that $|x||x^*| = 1$. Thus, as $x \rightarrow 0$, $u_\lambda^*(x) \rightarrow \phi_\lambda^{d-2}$. Next,

$$\begin{aligned} \frac{1}{2}h(0) &= \lim_{x \rightarrow 0}[u_\lambda(0) - u_\lambda^*(0)] = \left(\frac{\phi_\lambda}{|\lambda|}\right)^{d-2} - \phi_\lambda^{d-2} = \phi_\lambda^{d-2} \left[\frac{1 - |\lambda|^{d-2}}{|\lambda|^{d-2}}\right] \\ &\leq \left(\frac{\phi_{j-1}}{|\lambda|}\right)^{d-2} (d-2) \left[K^{-(j-1)} + O(K^{-2j})\right]. \end{aligned}$$

Thus, for sufficiently large j ,

$$h(0) \leq 4K(d-2) \left(\frac{\phi_{j-1}}{1 - K^{-(j-1)}}\right)^{d-2} K^{-j}.$$

Therefore,

$$\begin{aligned} \omega(0, \partial\mathcal{B}_r; \Omega_r) &\leq \sum_{j=n_r}^{\infty} \frac{2^{dd}K^2}{\epsilon^d} K^{(d-1)j} 4K(d-2) \left(\frac{\phi_{j-1}}{1 - K^{-(j-1)}}\right)^{d-2} K^{-j} \\ &\leq \frac{2^{d+2}d(d-2)K^{2d-1}}{\epsilon^d(K-1)^{d-2}} \sum_{j=n_r}^{\infty} (\phi_{j-1} K^{j-1})^{d-2} < 1 \end{aligned}$$

provided n_r is suitably selected as described at the start of the proof. Thus, $\omega(0, \partial\mathcal{B}_r; \Omega_r) < 1$ and hence we see that $\omega(0, \partial\mathbb{B}; \Omega) > 0$ as required.

3.2. Avoidability implies the integral condition (1.1). Now we assume that $\omega(0, \partial\mathbb{B}; \Omega) > 0$ and we'll show (1.1) holds. We begin by ignoring all obstacles with centres in an annulus A_j where j is odd. We let

$$\Omega' = \mathbb{B} \setminus \bigcup_{\lambda \in A_j, j \text{ even}} B(\lambda, r_\lambda)$$

and note that since $\omega(0, \partial\mathbb{B}; \Omega) > 0$, then $\omega(0, \partial\mathbb{B}; \Omega') > 0$. We choose $K > \max\{4, \frac{1+R}{1-R}\}$, where R is the constant mentioned in the definition of regularly spaced. We let P_j denote the probability that Brownian motion starting at the origin hits S_{j+1} before hitting any of the obstacles with centres in B_j but not in any A_i where i is odd. We let Q_j denote the supremum of the probabilities that Brownian motion starting on S_{j-1} hits S_{j+1} before hitting any of the obstacles with centres in A_j . We note that $P_j \leq Q_j P_{j-2}$ and that therefore for n even

$$P_n \leq P_0 \prod_{j=1, j \text{ even}}^n Q_j.$$

Since $\omega(0, \partial\mathbb{B}; \Omega') = \delta > 0$, it follows that $P_n \geq \delta$ for all n and, since $Q_j < 1$,

$$(3.2) \quad \sum_{j=1, j \text{ even}}^{\infty} (1 - Q_j) < \infty.$$

We note that $1 - Q_j$ is the infimum over $x \in S_{j-1}$ of the probability that Brownian motion starting at x hits a ball with centre in A_j before hitting S_{j+1} . Thus, if we consider only a single ball near x , say B_{λ_x} where λ_x is the centre of the ball near x as described in Lemma 3.1, then

$$1 - Q_j \geq \inf_{x \in S_{j-1}} \omega(x, \partial B_{\lambda_x}; B_{j+1} \setminus B_{\lambda_x}).$$

Thus, we need a lower bound for $\omega(x, \partial B_{\lambda_x}; B_{j+1} \setminus B_{\lambda_x})$. We want a suitable function h_j that is harmonic on $B_{j+1} \setminus B_{\lambda_x}$, continuous on its closure and also satisfies $h_j(y) \leq 1$, $y \in \partial B_{\lambda_x}$ and $h_j(y) \leq 0$, $y \in S_{j+1}$. Then we can again avail of the Maximum Principle to obtain the required lower bound. Consider the function

$$h_j(y) = u_\lambda(y) - u_\lambda^*(y),$$

where

$$u_\lambda(y) = \left[\frac{\phi_\lambda}{|y - \lambda_x|} \right]^{d-2}, \quad u_\lambda^*(y) = \left(\frac{\rho_{j+1}}{|y|} \right)^{d-2} \left[\frac{\phi_\lambda}{|y^* - \lambda_x|} \right]^{d-2} \text{ and } y^* = \frac{\rho_{j+1}^2}{|y|^2} y.$$

Then $h_j(y)$ satisfies the required criteria as both u_λ and u_λ^* are harmonic, $h_j \leq u_\lambda = 1$ on ∂B_{λ_x} , and $u_\lambda = u_\lambda^*$ on S_{j+1} . Next, we want a lower estimate for h_j at the point $x \in S_{j-1}$. With the help of Lemma 3.1,

$$\begin{aligned} u_\lambda(x) - u_\lambda^*(x) &= \left[\frac{\phi_\lambda}{|x - \lambda_x|} \right]^{d-2} - \left(\frac{\rho_{j+1}}{\rho_{j-1}} \right)^{d-2} \left[\frac{\phi_\lambda}{|x^* - \lambda_x|} \right]^{d-2} \\ &\geq \left(\frac{\phi_j}{|x - \lambda_x|} \right)^{d-2} \left[1 - \left(\frac{\rho_{j+1}}{D\rho_{j-1}} \right)^{d-2} \right], \end{aligned}$$

where $D = K/(K-1) > 1$. Then for sufficiently large j , namely j where

$$\frac{\rho_{j+1}}{\rho_{j-1}} < \frac{1+D}{2},$$

we find that

$$u_\lambda(x) - u_\lambda^*(x) \geq c \left(\frac{\phi_j}{|x - \lambda_x|} \right)^{d-2},$$

where c is some positive constant.

By (3.1), we find that for $x \in S_{j-1}$,

$$\omega(x, \partial B_{\lambda_x}; B_{j+1} \setminus B_{\lambda_x}) \geq h_j(x) = u_\lambda(x) - u_\lambda^*(x) \geq c(K-1)^{2-d} (\phi_j K^j)^{d-2}.$$

It now follows from (3.2) that

$$\sum_{j=1, j \text{ even}}^{\infty} (\phi_j K^j)^{d-2} < \infty.$$

Similarly it may be shown that

$$\sum_{j=1, j \text{ odd}}^{\infty} (\phi_j K^j)^{d-2} < \infty,$$

and so

$$\sum_{j=1}^{\infty} (\phi_j K^j)^{d-2} < \infty.$$

Hence, (1.1) holds and the proof is complete.

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